ME 141

Engineering Mechanics

Lecture 9: Kinematics of particles

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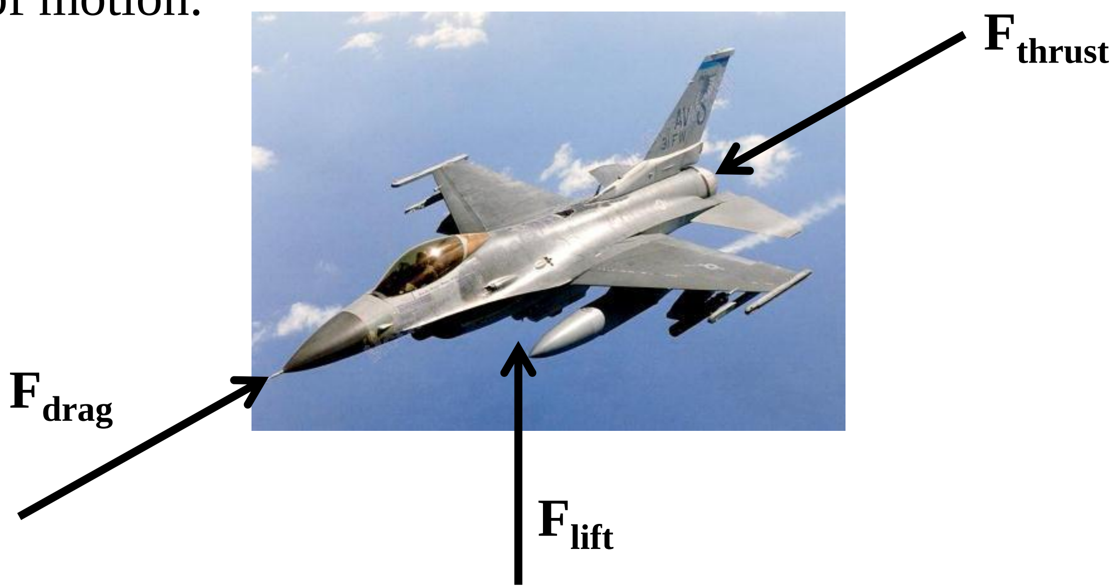
Courtesy: Vector Mechanics for Engineers, Beer and Johnston

Introduction

- **Dynamics includes:**

Kinematics: study of the geometry of motion.
Relates displacement, velocity, acceleration, and time ***without reference*** to the cause of motion.



Kinetics: study of the relations existing between the forces acting on a body, the mass of the body, and the motion of the body. Kinetics is used to predict the motion caused by given forces or to determine the forces required to produce a given motion.

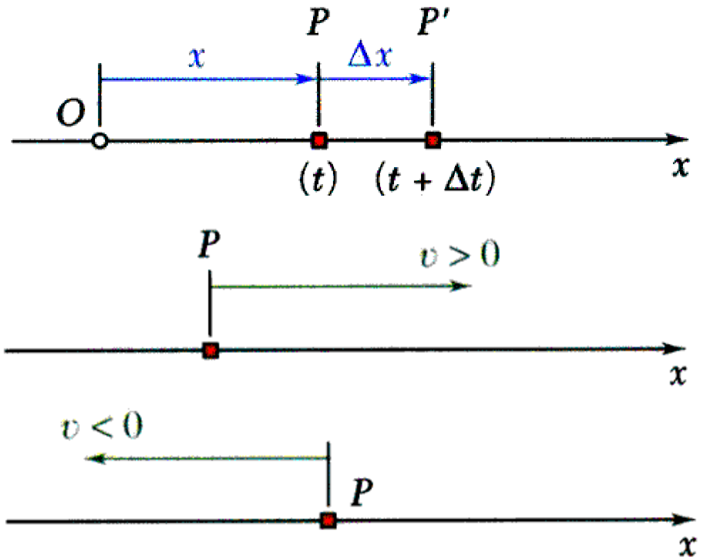
Introduction

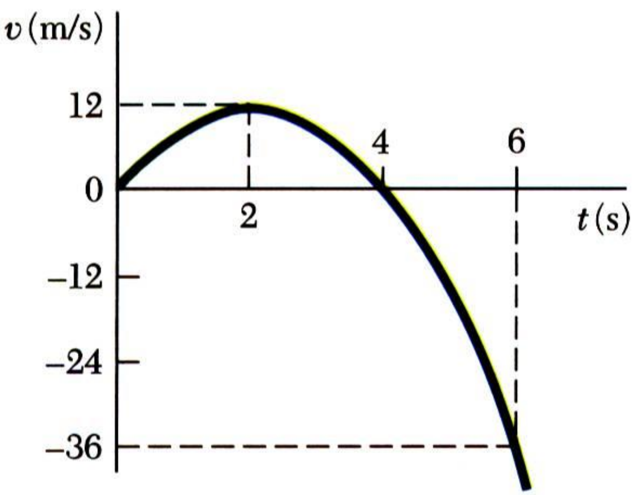
- **Particle kinetics includes:**

  - ***<u>Rectilinear motion</u>***: position, velocity, and acceleration of a particle as it moves along a straight line.

- ***<u>Curvilinear motion</u>***: position, velocity, and acceleration of a particle as it moves along a curved line in two or three dimensions.

Rectilinear Motion: Position, Velocity & Acceleration





- Consider particle which occupies position *P* at time *t* and *P'* at *t+Δt*,

$$Average\ velocity = \frac{\Delta x}{\Delta t}$$

$$Instantaneous\ velocity = v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$$

- Instantaneous velocity may be positive or negative. Magnitude of velocity is referred to as *particle speed.*

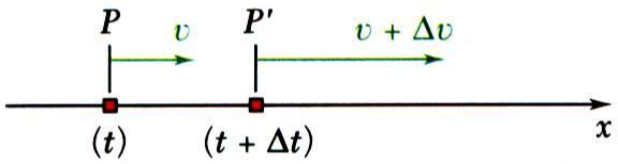
- From the definition of a derivative,

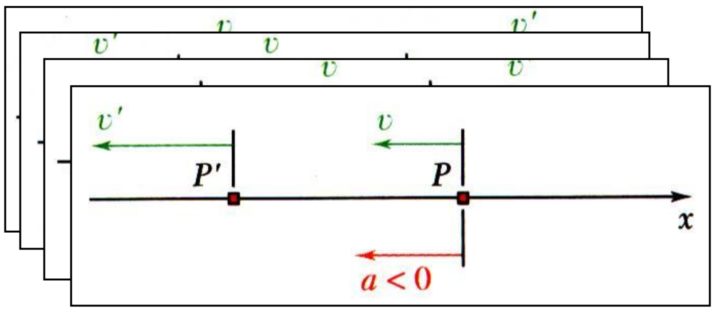
$$\boxed{v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}}$$

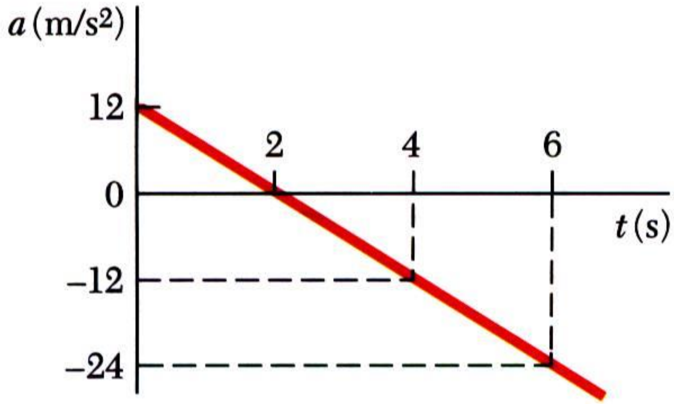
e.g., $x = 6t^2 - t^3$

$$v = \frac{dx}{dt} = 12t - 3t^2$$

Rectilinear Motion: Position, Velocity & Acceleration







- Consider particle with velocity v at time t and v' at $t+\Delta t$,

$$\textit{Instantaneous acceleration} = a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t}$$

- Instantaneous acceleration may be:
  - positive: increasing positive velocity or decreasing negative velocity
  - negative: decreasing positive velocity or increasing negative velocity.

- From the definition of a derivative,

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt} = \frac{d^2 x}{dt^2}$$

e.g. $v = 12t - 3t^2$

$$a = \frac{dv}{dt} = 12 - 6t$$

Determination of the Motion of a Particle

- **We often determine accelerations from the forces applied (kinetics will be covered later)**

- **Generally have three classes of motion**
  - acceleration given as a function of *time*, $a = f(t)$
  - acceleration given as a function of *position*, $a = \mathrm{f}(x)$
  - acceleration given as a function of *velocity*, $a = \mathrm{f}(v)$

- **Can you think of a physical example of when force is a function of position? When force is a function of velocity?**

a spring

drag

Acceleration as a function of time, position, or velocity

If….	Kinematic relationship	Integrate
$a = a(t)$	$\dfrac{dv}{dt} = a(t)$	$\displaystyle\int_{v_0}^{v} dv = \int_0^t a(t)\,dt$
$a = a(x)$	$dt = \dfrac{dx}{v}$ and $a = \dfrac{dv}{dt}$ $\downarrow$ $v\,dv = a(x)\,dx$	$\displaystyle\int_{v_0}^{v} v\,dv = \int_{x_0}^{x} a(x)\,dx$
$a = a(v)$	$\dfrac{dv}{dt} = a(v)$	$\displaystyle\int_{v_0}^{v} \frac{dv}{a(v)} = \int_0^t dt$
	$v\dfrac{dv}{dx} = a(v)$	$\displaystyle\int_{x_0}^{x} dx = \int_{v_0}^{v} \frac{v\,dv}{a(v)}$

Uniform Rectilinear Motion

During free-fall, a parachutist reaches terminal velocity when her weight equals the drag force. If motion is in a straight line, this is uniform rectilinear motion.

For a particle in uniform rectilinear motion, the acceleration is zero and the velocity is constant.

$$\frac{dx}{dt} = v = \text{constant}$$

$$\int_{x_0}^{x} dx = v \int_{0}^{t} dt$$

$$x - x_0 = vt$$

$$x = x_0 + vt$$

Careful – these only apply to uniform rectilinear motion!

Uniformly Accelerated Rectilinear Motion

If forces applied to a body are constant (and in a constant direction), then you have uniformly accelerated rectilinear motion.

Another example is free-fall when drag is negligible

Uniformly Accelerated Rectilinear Motion

For a particle in uniformly accelerated rectilinear motion, the acceleration of the particle is constant. You may recognize these constant acceleration equations from your physics courses.

$$\frac{dv}{dt} = a = \text{constant} \qquad \int_{v_0}^{v} dv = a\int_0^t dt \qquad v = v_0 + at$$

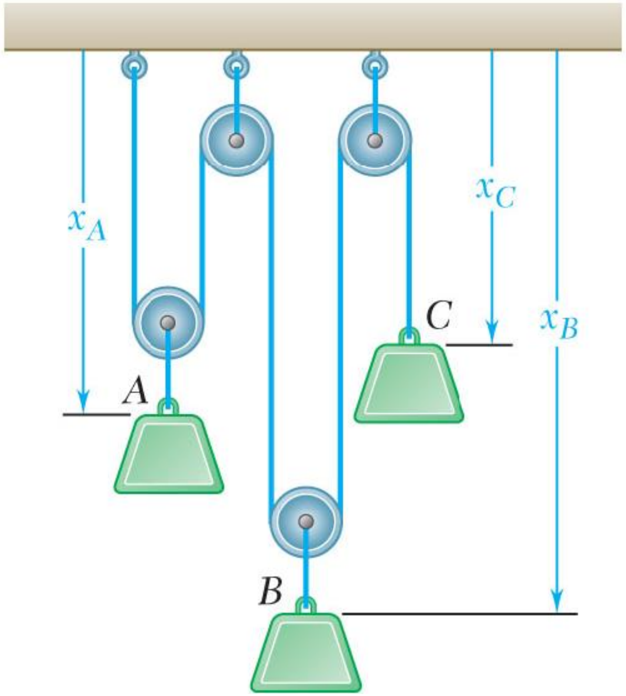
$$\frac{dx}{dt} = v_0 + at \qquad \int_{x_0}^{x} dx = \int_0^t \left(v_0 + at\right) dt \qquad x = x_0 + v_0 t + \tfrac{1}{2} a t^2$$

$$v\frac{dv}{dx} = a = \text{constant} \qquad \int_{v_0}^{v} v\, dv = a\int_{x_0}^{x} dx \qquad v^2 = v_0^2 + 2a\left(x - x_0\right)$$

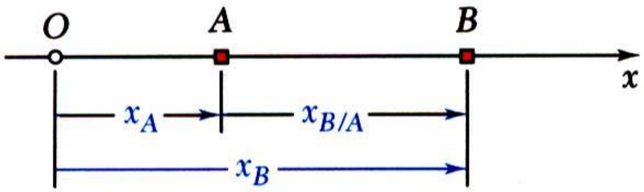
Careful – these only apply to uniformly accelerated rectilinear motion!

Motion of Several Particles

We may be interested in the motion of several different particles, whose motion may be independent or linked together.



Motion of Several Particles: Relative Motion



- For particles moving along the same line, time should be recorded from the same starting instant and displacements should be measured from the same origin in the same direction.

$$x_{B/A} = x_B - x_A = \text{ relative position of } B \text{ with respect to } A$$

$$x_B = x_A + x_{B/A}$$

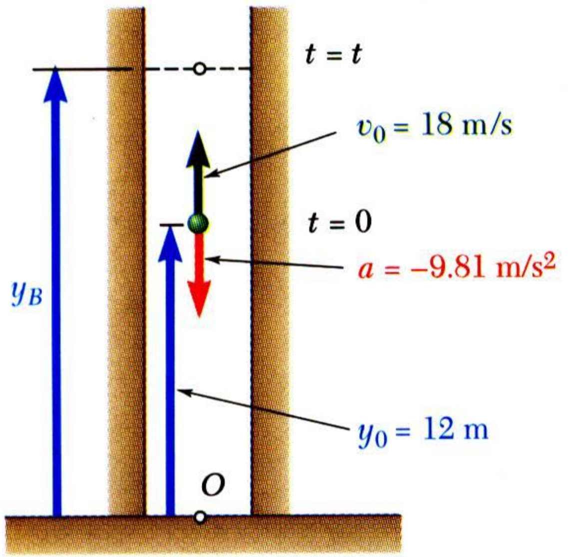
$$v_{B/A} = v_B - v_A = \text{ relative velocity of } B \text{ with respect to } A$$

$$v_B = v_A + v_{B/A}$$

$$a_{B/A} = a_B - a_A = \text{ relative acceleration of } B \text{ with respect to } A$$

$$a_B = a_A + a_{B/A}$$

Sample Problem 11.4



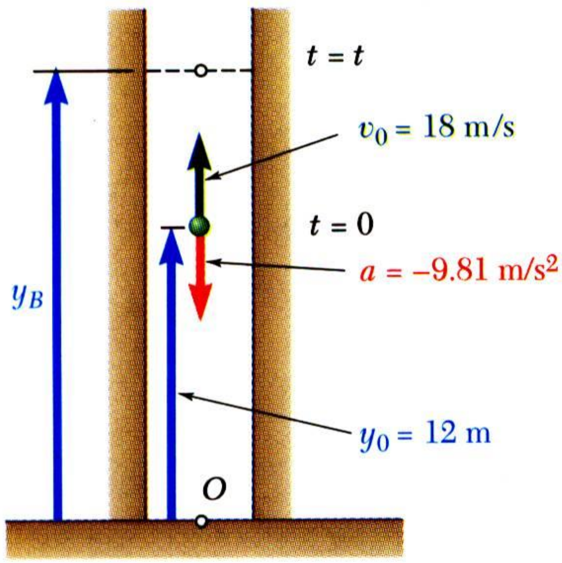
Ball thrown vertically from 12 m level in elevator shaft with initial velocity of 18 m/s. At same instant, open-platform elevator passes 5 m level moving upward at 2 m/s.

Determine *(a)* when and where ball hits elevator and *(b)* relative velocity of ball and elevator at contact.

SOLUTION:

- Substitute initial position and velocity and constant acceleration of ball into general equations for uniformly accelerated rectilinear motion.
- Substitute initial position and constant velocity of elevator into equation for uniform rectilinear motion.
- Write equation for relative position of ball with respect to elevator and solve for zero relative position, i.e., impact.
- Substitute impact time into equation for position of elevator and relative velocity of ball with respect to elevator.

Sample Problem 11.4

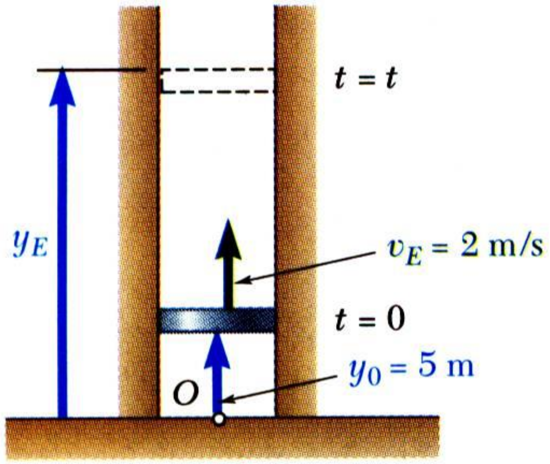


SOLUTION:

- Substitute initial position and velocity and constant acceleration of ball into general equations for uniformly accelerated rectilinear motion.

$$v_B = v_0 + at = 18\frac{\text{m}}{\text{s}} - \left(9.81\frac{\text{m}}{\text{s}^2}\right)t$$

$$y_B = y_0 + v_0 t + \tfrac{1}{2}at^2 = 12\,\text{m} + \left(18\frac{\text{m}}{\text{s}}\right)t - \left(4.905\frac{\text{m}}{\text{s}^2}\right)t^2$$

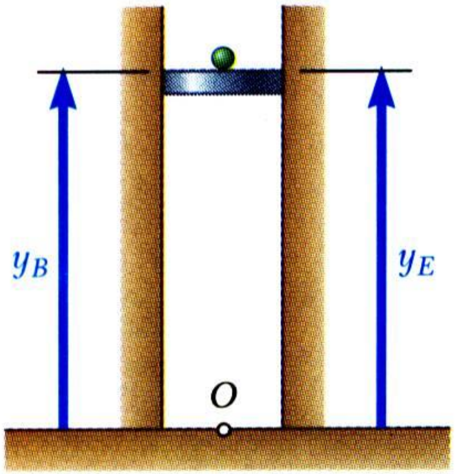


- Substitute initial position and constant velocity of elevator into equation for uniform rectilinear motion.

$$v_E = 2\frac{\text{m}}{\text{s}}$$

$$y_E = y_0 + v_E t = 5\,\text{m} + \left(2\frac{\text{m}}{\text{s}}\right)t$$

Sample Problem 11.4



- Write equation for relative position of ball with respect to elevator and solve for zero relative position, i.e., impact.

$$y_{B/E} = \left(12 + 18t - 4.905t^2\right) - \left(5 + 2t\right) = 0$$

$$\boxed{\begin{aligned} t &= -0.39\,\mathrm{s} \quad (\text{meaningless}) \\ t &= 3.65\,\mathrm{s} \end{aligned}}$$

- Substitute impact time into equations for position of elevator and relative velocity of ball with respect to elevator.

$$y_E = 5 + 2(3.65)$$

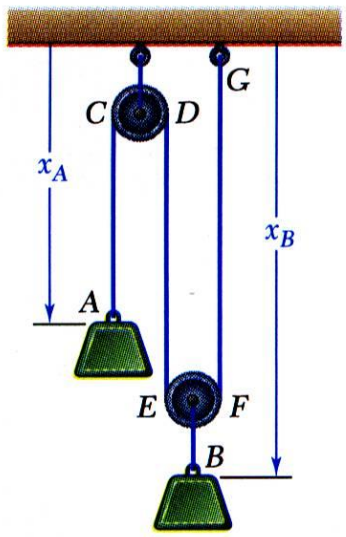
$$\boxed{y_E = 12.3\,\mathrm{m}}$$

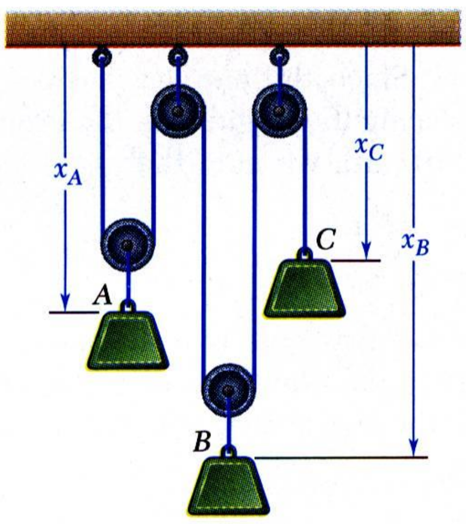
$$v_{B/E} = \left(18 - 9.81t\right) - 2$$

$$= 16 - 9.81(3.65)$$

$$\boxed{v_{B/E} = -19.81\,\frac{\mathrm{m}}{\mathrm{s}}}$$

Motion of Several Particles: Dependent





- Position of a particle may *depend* on position of one or more other particles.

- Position of block B depends on position of block A. Since rope is of constant length, it follows that sum of lengths of segments must be constant.

$$\boxed{x_A + 2x_B =} \text{ constant (one degree of freedom)}$$

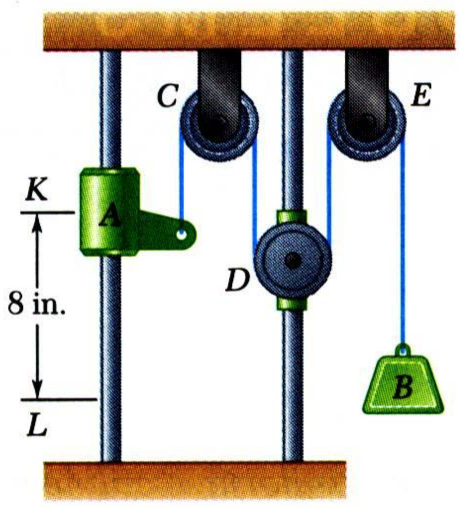
- Positions of three blocks are dependent.

$$\boxed{2x_A + 2x_B + x_C =} \text{ constant (two degrees of freedom)}$$

- For linearly related positions, similar relations hold between velocities and accelerations.

$$\boxed{\begin{array}{l} 2\dfrac{dx_A}{dt} + 2\dfrac{dx_B}{dt} + \dfrac{dx_C}{dt} = 0 \quad \text{or} \quad 2v_A + 2v_B + v_C = 0 \\ 2\dfrac{dv_A}{dt} + 2\dfrac{dv_B}{dt} + \dfrac{dv_C}{dt} = 0 \quad \text{or} \quad 2a_A + 2a_B + a_C = 0 \end{array}}$$

Sample Problem 11.5

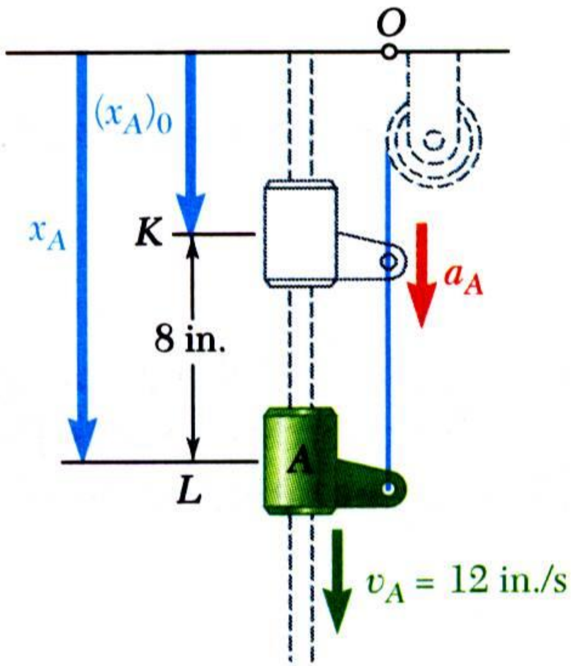


Pulley D is attached to a collar which is pulled down at 3 in./s. At $t = 0$, collar A starts moving down from K with constant acceleration and zero initial velocity. Knowing that velocity of collar A is 12 in./s as it passes L, determine the change in elevation, velocity, and acceleration of block B when block A is at L.

SOLUTION:

- Define origin at upper horizontal surface with positive displacement downward.
- Collar A has uniformly accelerated rectilinear motion. Solve for acceleration and time t to reach L.
- Pulley D has uniform rectilinear motion. Calculate change of position at time t.
- Block B motion is dependent on motions of collar A and pulley D. Write motion relationship and solve for change of block B position at time t.
- Differentiate motion relation twice to develop equations for velocity and acceleration of block B.

Sample Problem 11.5



SOLUTION:

- Define origin at upper horizontal surface with positive displacement downward.

- Collar *A* has uniformly accelerated rectilinear motion. Solve for acceleration and time *t* to reach *L*.

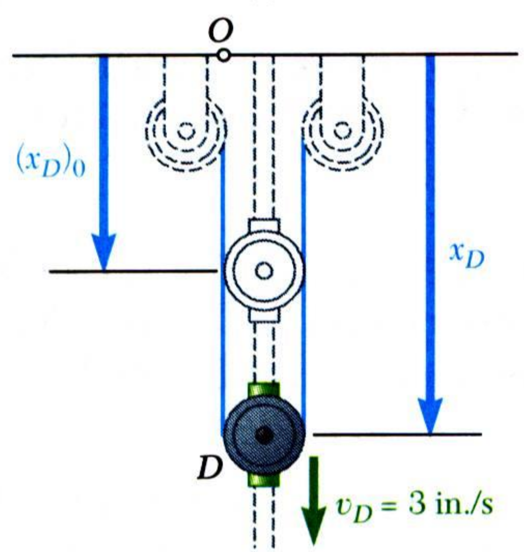
$$v_A^2 = (v_A)_0^2 + 2a_A\left[x_A - (x_A)_0\right]$$

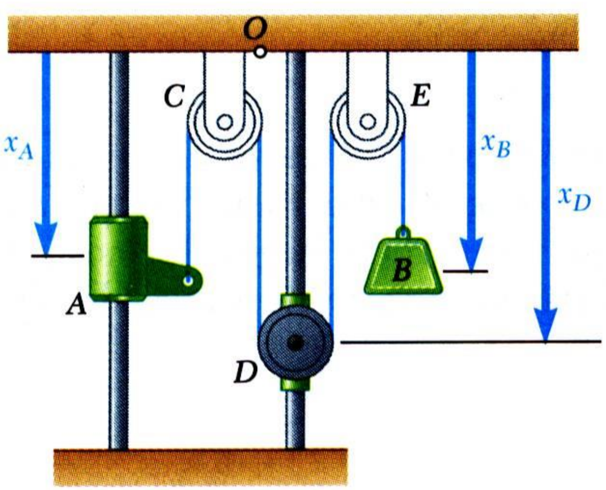
$$\left(12\,\frac{\text{in.}}{\text{s}}\right)^2 = 2a_A(8\,\text{in.}) \qquad a_A = 9\,\frac{\text{in.}}{\text{s}^2}$$

$$v_A = (v_A)_0 + a_A t$$

$$12\,\frac{\text{in.}}{\text{s}} = 9\,\frac{\text{in.}}{\text{s}^2}\,t \qquad t = 1.333\,\text{s}$$

Sample Problem 11.5





- Pulley *D* has uniform rectilinear motion. Calculate change of position at time *t*.

$$x_D = (x_D)_0 + v_D t$$

$$x_D - (x_D)_0 = \left(3\frac{\text{in.}}{\text{s}}\right)(1.333\,\text{s}) = 4\,\text{in.}$$

- Block *B* motion is dependent on motions of collar *A* and pulley *D*. Write motion relationship and solve for change of block *B* position at time *t*.

Total length of cable remains constant,

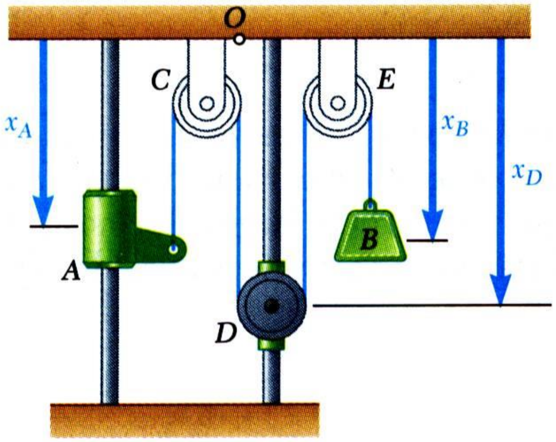
$$x_A + 2x_D + x_B = (x_A)_0 + 2(x_D)_0 + (x_B)_0$$

$$[x_A - (x_A)_0] + 2[x_D - (x_D)_0] + [x_B - (x_B)_0] = 0$$

$$(8\,\text{in.}) + 2(4\,\text{in.}) + [x_B - (x_B)_0] = 0$$

$$\boxed{x_B - (x_B)_0 = -16\,\text{in.}}$$

Sample Problem 11.5



- Differentiate motion relation twice to develop equations for velocity and acceleration of block *B*.

$$x_A + 2x_D + x_B = \text{constant}$$

$$v_A + 2v_D + v_B = 0$$

$$\left(12\frac{\text{in.}}{\text{s}}\right) + 2\left(3\frac{\text{in.}}{\text{s}}\right) + v_B = 0$$

$$\boxed{v_B = -18\frac{\text{in.}}{\text{s}}}$$

$$a_A + 2a_D + a_B = 0$$

$$\left(9\frac{\text{in.}}{\text{s}^2}\right) + v_B = 0$$

$$\boxed{a_B = -9\frac{\text{in.}}{\text{s}^2}}$$

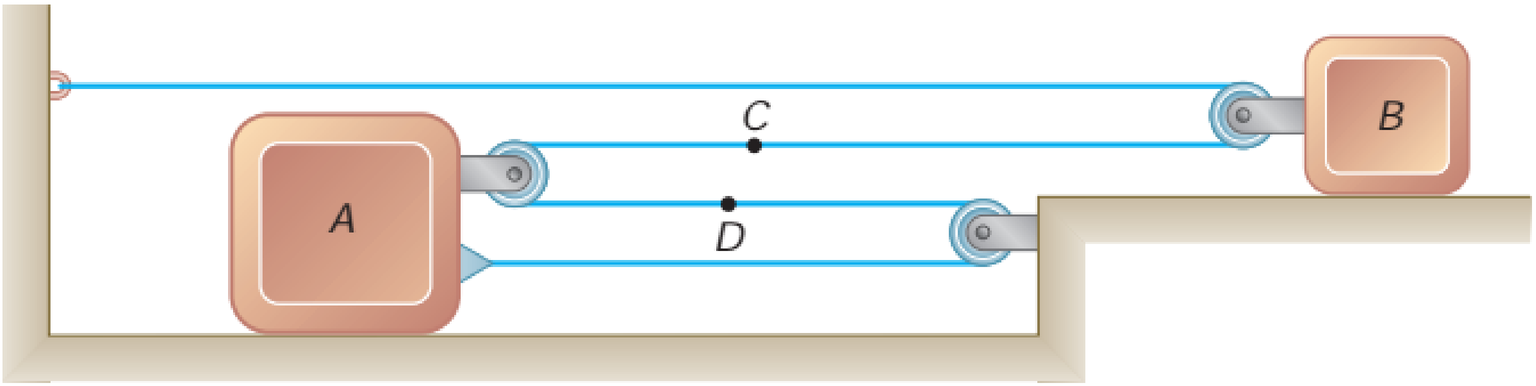
Prob # 11.52

At the instant shown, slider block *B* is moving with a constant acceleration, and its speed is 150 mm/s. Knowing that after slider block *A* has moved 240 mm to the right its velocity is 60 mm/s,

determine (*a*) the accelerations of *A* and *B*,

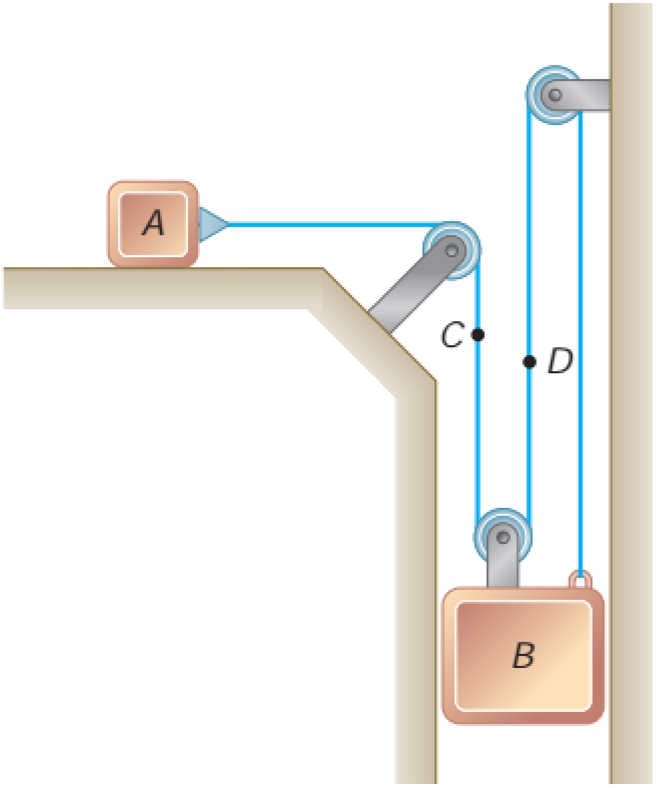
(*b*) the acceleration of portion *D* of the cable,

(*c*) the velocity and the change in position of slider block *B* after 4 s.



Prob # 11.49

Slider block *A* moves to the left with a constant velocity of 6 m/s.
Determine
(*a*) the velocity of block *B*,
(*b*) the velocity of portion *D* of the cable,
(*c*) the relative velocity of portion *C* of the cable with respect to portion *D*.



Curvilinear Motion: Position, Velocity & Acceleration

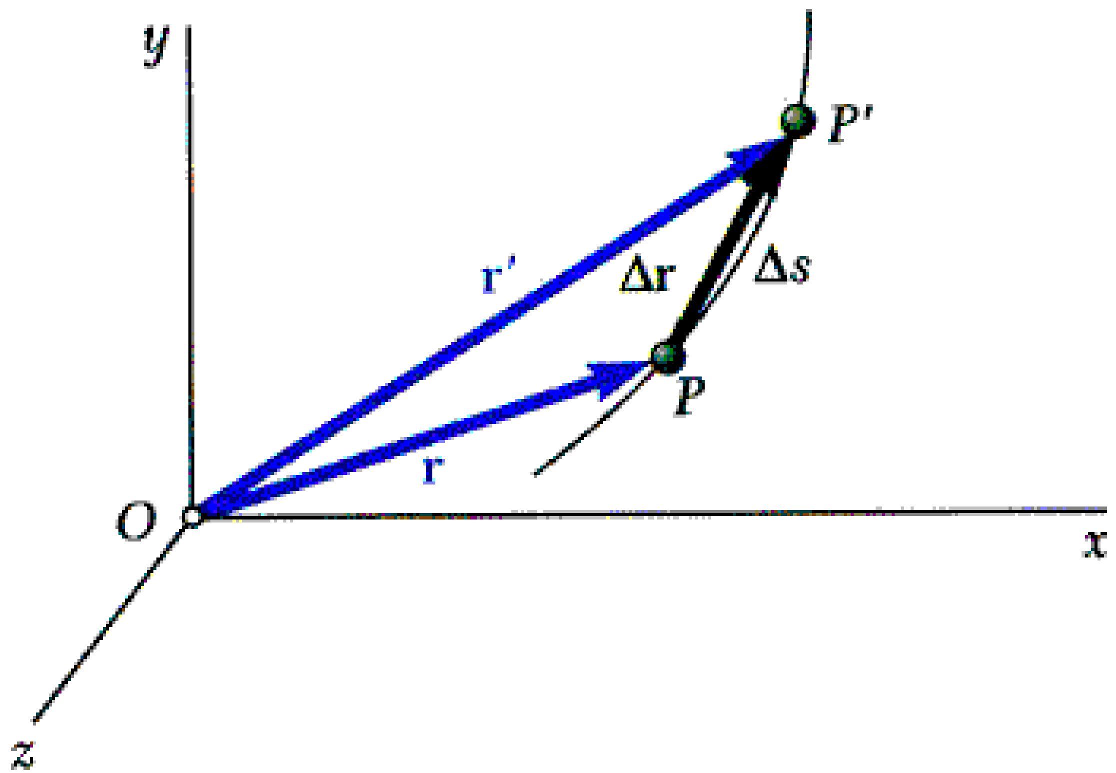
The softball and the car both undergo curvilinear motion.

- A particle moving along a curve other than a straight line is in *curvilinear motion*.

Curvilinear Motion: Position, Velocity & Acceleration

- The *position vector* of a particle at time *t* is defined by a vector between origin *O* of a fixed reference frame and the position occupied by particle.

- Consider a particle which occupies position *P* defined by $\vec{r}$ at time *t* and *P'* defined by $\vec{r}\,'$ at $t + \Delta t$,



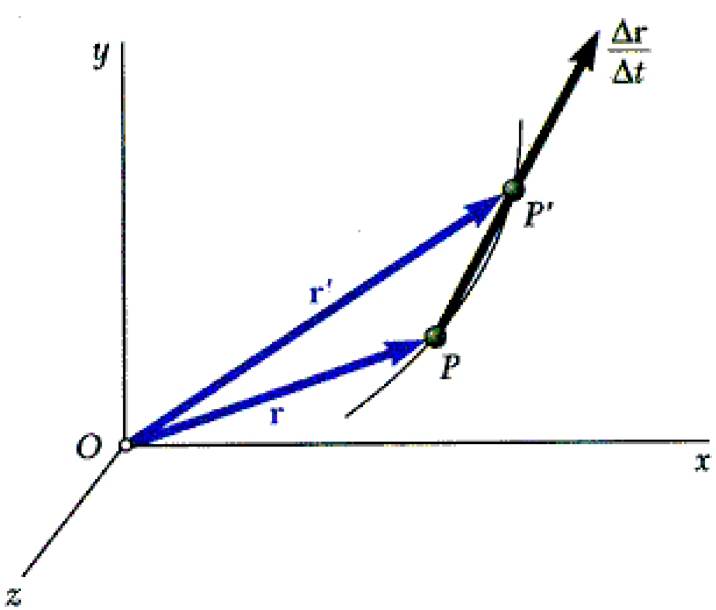
Curvilinear Motion: Position, Velocity & Acceleration

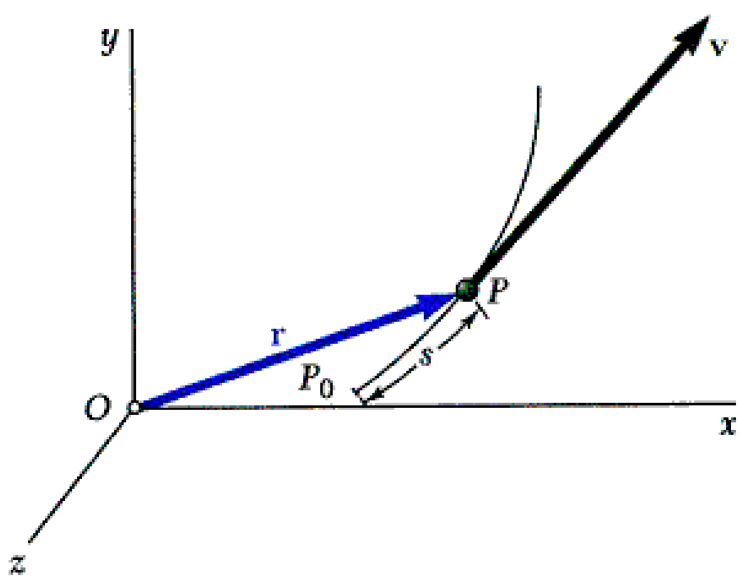
Instantaneous velocity
(vector)

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}$$

Instantaneous speed
(scalar)

$$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}$$

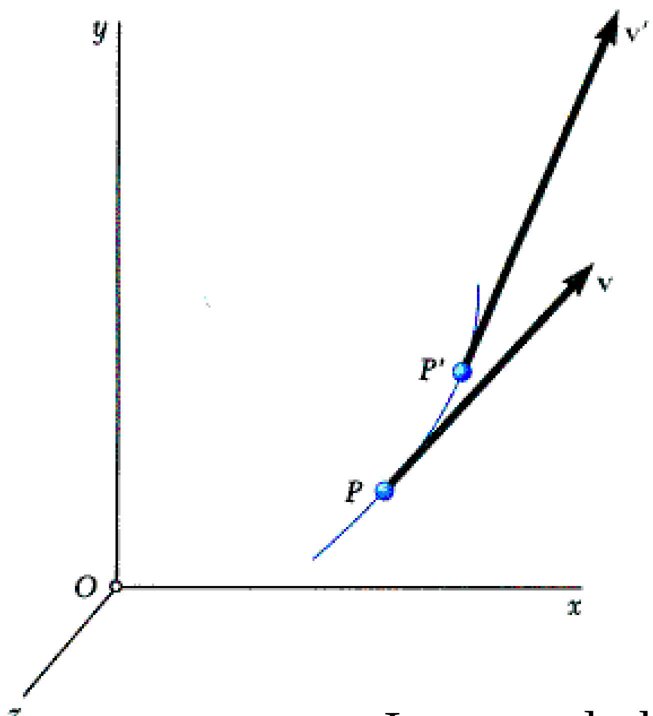


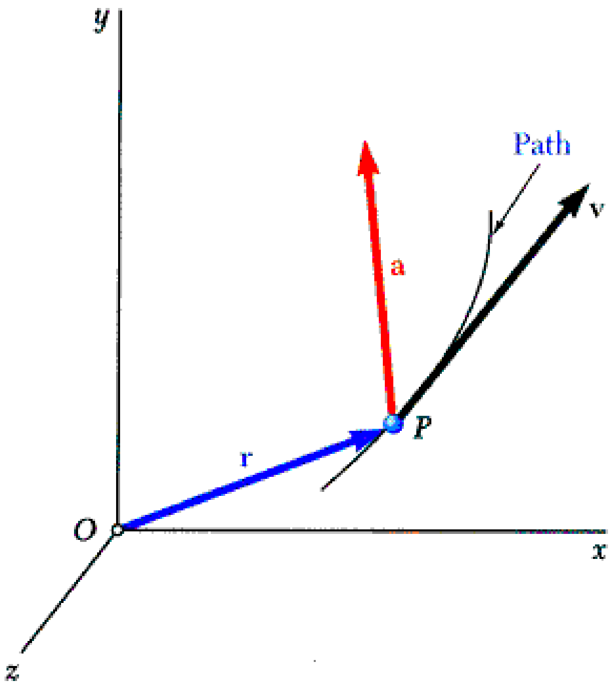


Curvilinear Motion: Position, Velocity & Acceleration

- Consider velocity $\vec{v}$ of a particle at time t and velocity $\vec{v}'$ at $t + \Delta t$,

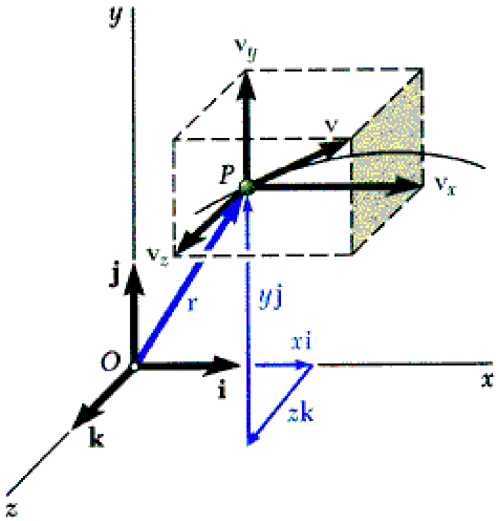
$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt} = \text{ instantaneous acceleration (vector)}$$

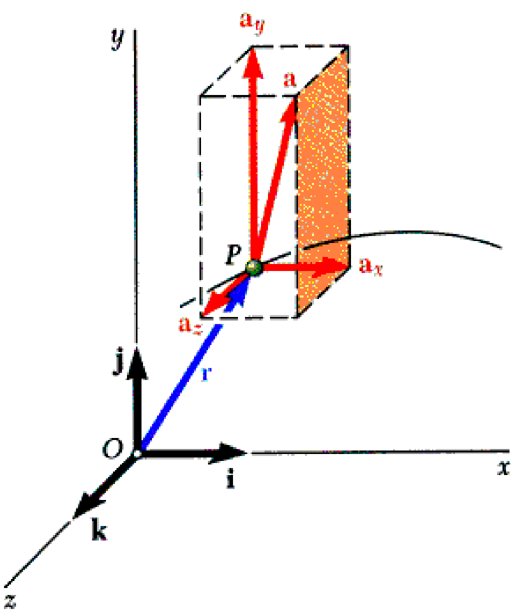




- In general, the acceleration vector is not tangent to the particle path and velocity vector.

Rectangular Components of Velocity & Acceleration





- When position vector of particle P is given by its rectangular components,

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

- Velocity vector,

$$\vec{v} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k} = \dot{x}\vec{i} + \dot{y}\vec{j} + \dot{z}\vec{k}$$

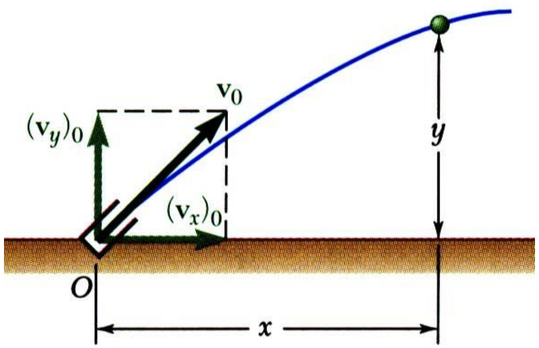
$$= v_x\vec{i} + v_y\vec{j} + v_z\vec{k}$$

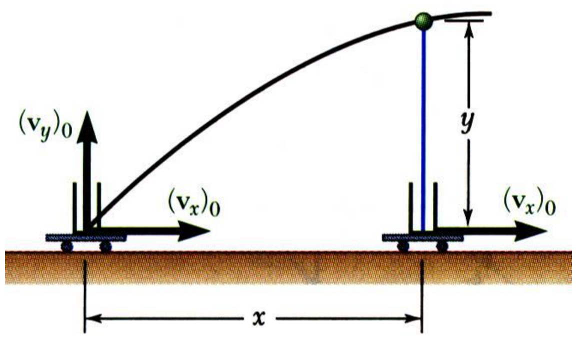
- Acceleration vector,

$$\vec{a} = \frac{d^2x}{dt^2}\vec{i} + \frac{d^2y}{dt^2}\vec{j} + \frac{d^2z}{dt^2}\vec{k} = \ddot{x}\vec{i} + \ddot{y}\vec{j} + \ddot{z}\vec{k}$$

$$= a_x\vec{i} + a_y\vec{j} + a_z\vec{k}$$

Rectangular Components of Velocity & Acceleration





- Rectangular components particularly effective when component accelerations can be integrated independently, e.g., motion of a projectile,

$$\boxed{a_x = \ddot{x} = 0 \qquad a_y = \ddot{y} = -g \qquad a_z = \ddot{z} = 0}$$

with initial conditions,

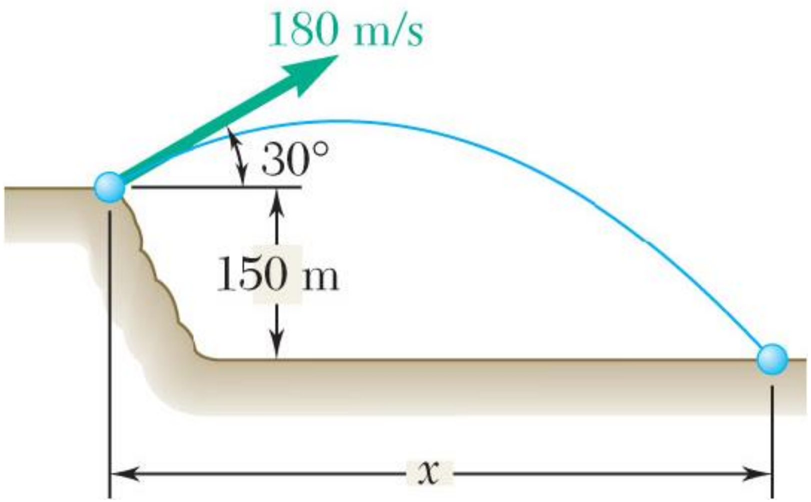
$$\boxed{x_0 = y_0 = z_0 = 0 \qquad (v_x)_0, (v_y)_0, (v_z)_0 = 0}$$

Integrating twice yields

$$v_x = (v_x)_0 \quad v_y = (v_y)_0 - gt \qquad v_z = 0$$
$$x = (v_x)_0 t \quad y = (v_y)_0 y - \tfrac{1}{2} gt^2 \quad z = 0$$

- Motion in horizontal direction is uniform.
- Motion in vertical direction is uniformly accelerated.
- Motion of projectile could be replaced by two independent rectilinear motions.

Sample Problem 11.7

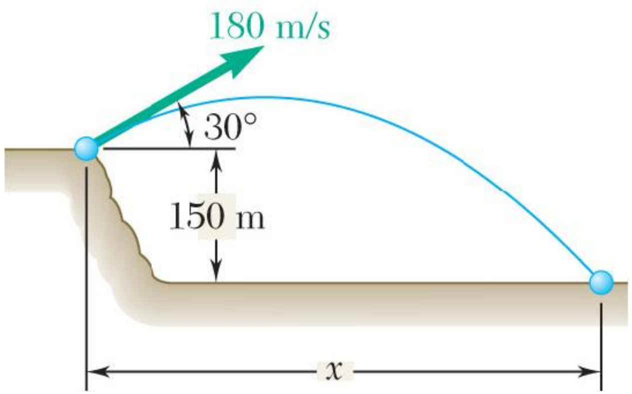


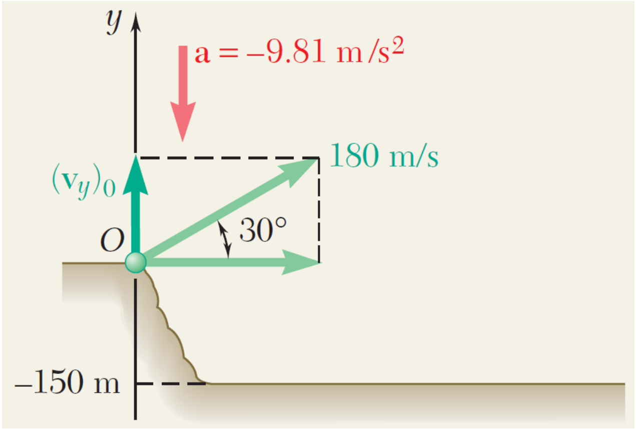
A projectile is fired from the edge of a 150-m cliff with an initial velocity of 180 m/s at an angle of 30° with the horizontal. Neglecting air resistance, find (*a*) the horizontal distance from the gun to the point where the projectile strikes the ground, (*b*) the greatest elevation above the ground reached by the projectile.

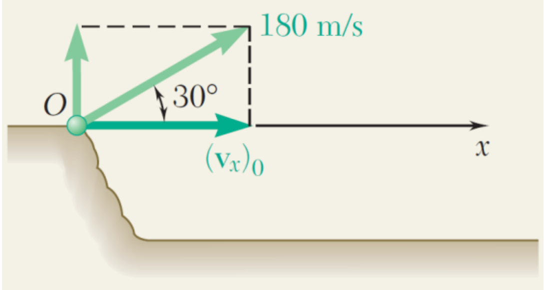
SOLUTION:

- Consider the vertical and horizontal motion separately (they are independent)
- Apply equations of motion in y-direction
- Apply equations of motion in x-direction
- Determine time *t* for projectile to hit the ground, use this to find the horizontal distance
- Maximum elevation occurs when $v_y=0$

Sample Problem 11.7







SOLUTION:

Given: $(v)_o = 180$ m/s $\quad (y)_o = 150$ m

$(a)_y = -9.81$ m/s² $\quad (a)_x = 0$ m/s²

Vertical motion – uniformly accelerated:

$$(v_y)_0 = (180 \text{ m/s}) \sin 30° = +90 \text{ m/s}$$

$$v_y = (v_y)_0 + at \qquad v_y = 90 - 9.81t \qquad (1)$$

$$y = (v_y)_0 t + \tfrac{1}{2}at^2 \qquad y = 90t - 4.90t^2 \qquad (2)$$

$$v_y^2 = (v_y)_0^2 + 2ay \qquad v_y^2 = 8100 - 19.62y \qquad (3)$$

Horizontal motion – uniform motion

Choose positive x to the right as shown

$$(v_x)_0 = (180 \text{ m/s}) \cos 30° = +155.9 \text{ m/s}$$

$$x = (v_x)_0 t \qquad x = 155.9t$$

Sample Problem 11.7

SOLUTION:

Horizontal distance

Projectile strikes the ground at: $y = -150 \text{ m}$

Substitute into equation (1) above

$$-150 = 90t - 4.90t^2$$

Solving for *t*, we take the positive root

$$t^2 - 18.37t - 30.6 = 0 \qquad t = 19.91 \text{ s}$$

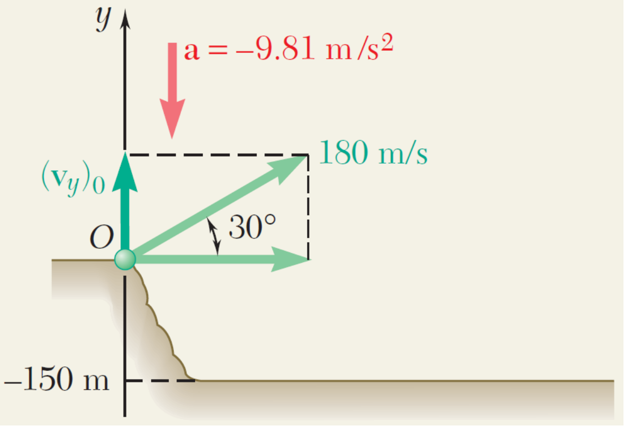
Substitute t into equation (4)

$$x = 155.9(19.91) \qquad x = 3100 \text{ m}$$

Maximum elevation occurs when $v_y = 0$

$$0 = 8100 - 19.62y \qquad y = 413 \text{ m}$$

Maximum elevation above the ground = $150 \text{ m} + 413 \text{ m} = 563 \text{ m}$

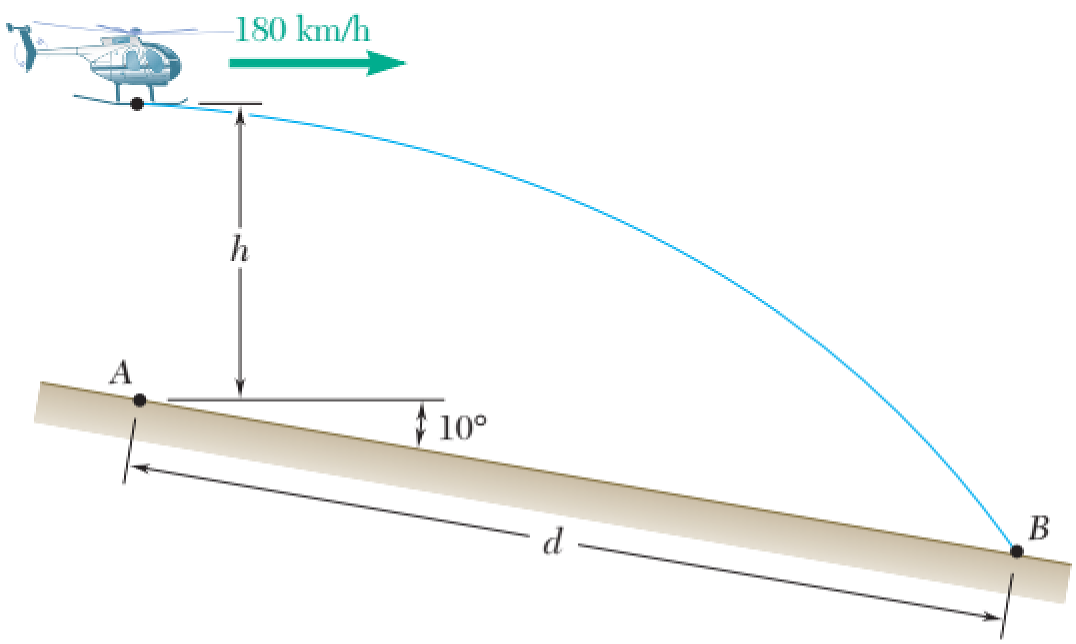


Prob # 11.98

A helicopter is flying with a constant horizontal velocity of 180 km/h and is directly above point *A* when a loose part begins to fall. The part lands 6.5 s later at point *B* on an inclined surface. Determine

(*a*) the distance *d* between points *A* and *B*.

(*b*) the initial height *h*

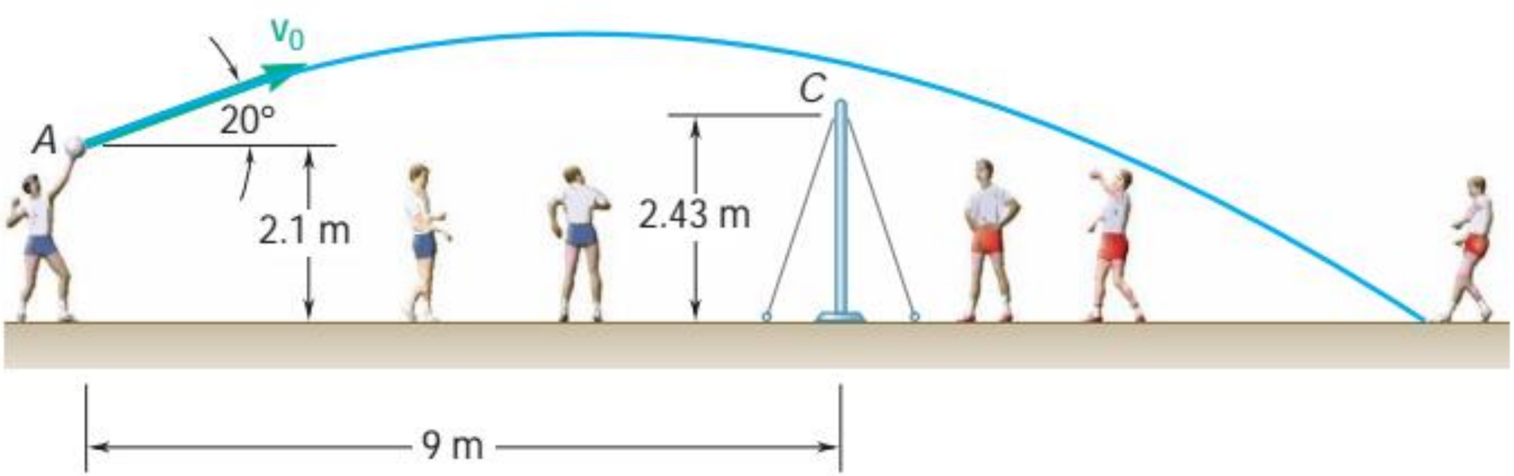


Prob # 11.103

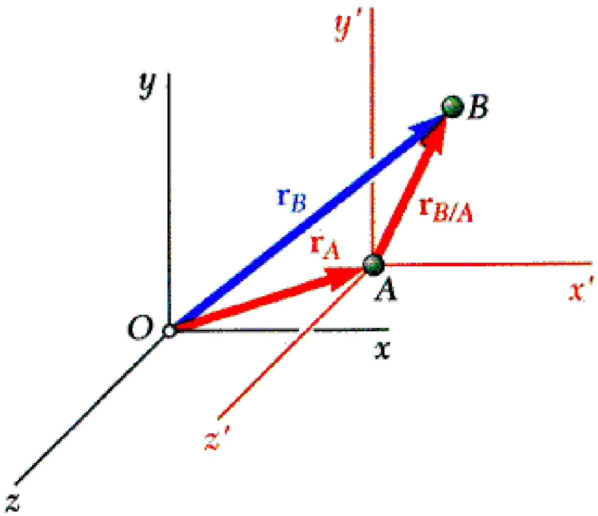
A volleyball player serves the ball with an initial velocity $\mathbf{v}_o$ of magnitude 13.40 m/s at an angle of 20° with the horizontal. Determine

(a) if the ball will clear the top of the net,

(b) how far from the net the ball will land.



Motion Relative to a Frame in Translation



- Designate one frame as the *fixed frame of reference*. All other frames not rigidly attached to the fixed reference frame are *moving frames of reference*.
- Position vectors for particles *A* and *B* with respect to the fixed frame of reference *Oxyz* are $\vec{r}_A$ and $\vec{r}_B$.
- Vector $\vec{r}_{B/A}$ joining *A* and *B* defines the position of *B* with respect to the moving frame *Ax'y'z'* and

$$\vec{r}_B = \vec{r}_A + \vec{r}_{B/A}$$

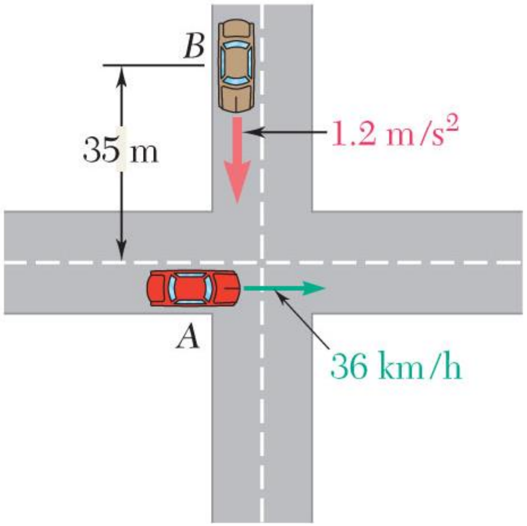
- Differentiating twice,

$\vec{v}_B = \vec{v}_A + \vec{v}_{B/A}$ $\quad \vec{v}_{B/A} =$ velocity of *B* relative to *A*.

$\vec{a}_B = \vec{a}_A + \vec{a}_{B/A}$ $\quad \vec{a}_{B/A} =$ acceleration of *B* relative to *A*.

- Absolute motion of *B* can be obtained by combining motion of *A* with relative motion of *B* with respect to moving reference frame attached to *A*.

Sample Problem 11.9

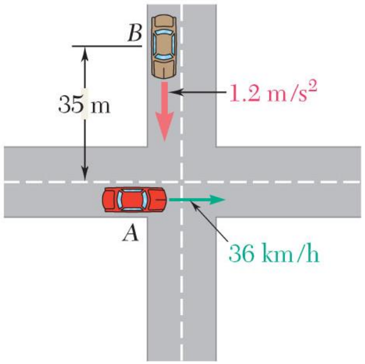


Automobile *A* is traveling east at the constant speed of 36 km/h. As automobile *A* crosses the intersection shown, automobile *B* starts from rest 35 m north of the intersection and moves south with a constant acceleration of 1.2 m/s^2. Determine the position, velocity, and acceleration of *B* relative to *A* 5 *s* after *A* crosses the intersection.

SOLUTION:

- Define inertial axes for the system
- Determine the position, speed, and acceleration of car A at $t = 5$ s
- Determine the position, speed, and acceleration of car B at $t = 5$ s
- Using vectors (Eqs 11.31, 11.33, and 11.34) or a graphical approach, determine the relative position, velocity, and acceleration

Sample Problem 11.9



SOLUTION:

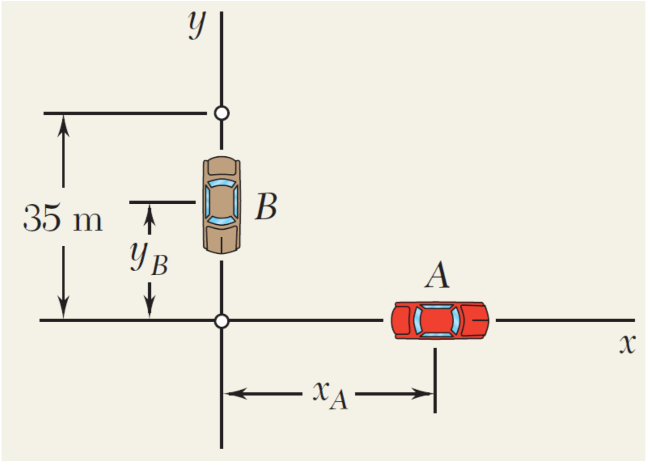
- Define axes along the road

Given: v_A=36 km/h, a_A= 0, $(x_A)_0$ = 0

$(v_B)_0$= 0, a_B= - 1.2 m/s², $(y_A)_0$ = 35 m

Determine motion of Automobile A:

$$v_A = \left(36\frac{\text{km}}{\text{h}}\right)\left(\frac{1000\text{ m}}{1\text{ km}}\right)\left(\frac{1\text{ h}}{3600\text{ s}}\right) = 10\text{ m/s}$$



We have uniform motion for A so:

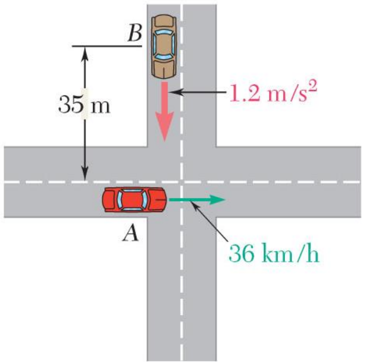
$$\begin{aligned} a_A &= 0 \\ v_A &= +10\text{ m/s} \\ x_A &= (x_A)_0 + v_A t = 0 + 10t \end{aligned}$$

At t = 5 s

$$\begin{aligned} a_A &= 0 \\ v_A &= +10\text{ m/s} \\ x_A &= +(10\text{ m/s})(5\text{ s}) = +50\text{ m} \end{aligned}$$

$$\begin{aligned} \mathbf{a}_A &= 0 \\ \mathbf{v}_A &= 10\text{ m/s} \rightarrow \\ \mathbf{r}_A &= 50\text{ m} \rightarrow \end{aligned}$$

Sample Problem 11.9

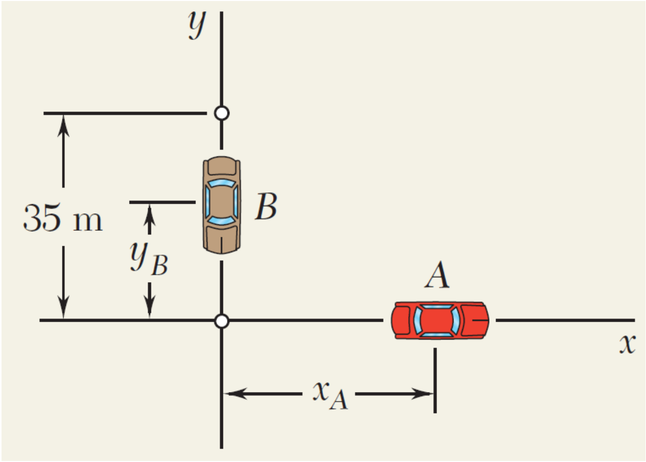


SOLUTION:

Determine motion of Automobile B:

We have uniform acceleration for B so:

$$a_B = -1.2 \text{ m/s}^2$$
$$v_B = (v_B)_0 + at = 0 - 1.2\ t$$
$$y_B = (y_B)_0 + (v_B)_0 t + \tfrac{1}{2} a_B t^2 = 35 + 0 - \tfrac{1}{2}(1.2)t^2$$



At $t = 5$ s

$$a_B = -1.2 \text{ m/s}^2$$
$$v_B = -(1.2 \text{ m/s}^2)(5 \text{ s}) = -6 \text{ m/s}$$
$$y_B = 35 - \tfrac{1}{2}(1.2 \text{ m/s}^2)(5 \text{ s})^2 = +20 \text{ m}$$

$$\mathbf{a}_B = 1.2 \text{ m/s}^2 \downarrow$$
$$\mathbf{v}_B = 6 \text{ m/s} \downarrow$$
$$\mathbf{r}_B = 20 \text{ m} \uparrow$$

Prob# 11.124

Knowing that at the instant shown block *A* has a velocity of 8 in./s and an acceleration of 6 in./s^2 both directed down the incline, determine

(a) the velocity of block *B*,

(b) the acceleration of block *B.*